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# Janus within Janus

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Abstract: We found a simple and interesting generalization of the non-supersymmetric Janus solution in type IIB string theory. The Janus solution can be thought of as a thick $AdS_d$-sliced domain wall in $AdS_{d+1}$ space. It turns out that the $AdS_d$-sliced domain wall can support its own $AdS_{d-1}$-sliced domain wall within it. Indeed this pattern persists further until it reaches the $AdS_2$-slice of the domain wall within self-similar $AdS_{p\,(2<p\leq d)}$-sliced domain walls. In other words the solution represents a sequence of little Janus nested in the interface of the parent Janus according to a remarkably simple "nesting" rule. Via the AdS/CFT duality, the dual gauge theory description is in general an interface CFT of higher codimensions.

Keywords: AdS-CFT Correspondence, Classical Theories of Gravity.



 

## Contents



## 1. Introduction

The non-supersymmetric backgrounds in string theory are often hard to control and it is not easy to make any definite statements with quantitative precision. Yet we would eventually have to deal with them, as our universe is not supersymmetric. Also for the purpose of understanding the confinement via the gauge theory/gravity duality, we hope to ultimately get a good handle on the non-supersymmetric circumstances.

A non-supersymmetric solution with reasonable tractability was found in [1] in the context of the AdS/CFT duality [2−5]. It is a dilatonic deformation of $AdS_5 \times S^5$ in type IIB supergravity and named Janus due to its rather curious characteristic. Several dilatonic deformations exist [6−9], but they typically lead to singular geometries. Although these solutions exhibit interesting phenomena suggesting possible gravity duals of confinement, the validity of their analyses is not too clear. In contrast the Janus solution is nonsingular, and the scalar curvature and string coupling can be kept small everywhere in spacetime. Furthermore the stability of the Janus solution was partially shown against the scalar field perturbations in [1] and remarkably later in [10] against a broad class of perturbations, suggesting that it is indeed stable.

Perhaps what makes Janus possibly interesting is its remarkable simplicity. Indeed in the dual gauge theory side, the Janus simply corresponds to having the different SYM coupling in each half of the boundary spacetime [1]. In other words, the SYM coupling jumps discontinuously when it moves from one half of the space to another, dividing the boundary spacetime into two characterized by two different values of the coupling constants — hence the name Janus, the god of gates, doors, doorways, beginnings, and endings in Roman mythology who is often symbolized by two faces. Two faces are joined at the interface. Although the 4-dimensional conformal symmetry SO(2, 4) is partially broken, the conformal symmetry SO(2, 3) on the interface is preserved. Hence the dual gauge theory is thought of as an interface CFT.[1] Remarkably the supergravity predictions were firmly confirmed by the dual interface CFT computations [11].

[1] An interface is similar to a defect. However, as emphasized in [11, 12], the former does not support any new degrees of freedom independent of those in the bulk, while the latter does. So we adopt the terminology interface CFT rather than defect CFT.



JHEP05(2006)031

Another aspect of the Janus solution is in its relation to the $AdS_4$ domain wall in $AdS_5$ [13] (see also an earlier work on the $AdS_4$ domain wall in 5D gauged supergravity [14]). By construction, the Janus can be thought of as a thick $AdS_4$-sliced domain wall in $AdS_5$. In [10] the Janus was generalized and studied by making use of the fake supergravity from the domain wall perspective. Some other aspects of the Janus as a domain wall were discussed in [15]. The brane configuration for the $AdS_4$ domain wall of [13] was proposed in [16]. In a similar spirit, one might hope to find a brane configuration for the Janus solution. The non-supersymmetric Janus clearly does not have any source or charges, so it does not seem to have such an interpretation. This lead to a search for the supersymmetric Janus, and in fact the supersymmetric generalization was found in [17] at the level of the 5D gauged supergravity and more recently in [18] in the full type IIB supergravity.

In this paper we study a generalization of the non-supersymmetric Janus in a different flavour.[2] The generalization is bound to complicate the original system. However, here it will be made with a remarkable simplicity for our new solution. The new solution in general represents a sequence of little Janus nested within the parent Janus — $AdS_2$-sliced Janus $\subset$ $AdS_3$-sliced Janus $\subset$ $AdS_4$-sliced Janus. This generalization follows a remarkably simple "nesting" rule. Indeed it is almost as simple as the original Janus.

In section 2, we review the Janus solution of [1] in type IIB supergravity. In section 3, we construct our new solution and propose its gauge theory dual description. In section 4, we end with brief discussions.

## 2. A review of the Janus solution

We focus on the simplest case of the Janus solution in type IIB string theory — a dilatonic deformation of $AdS_5 \times S^5$ found in [1]. In this case only the metric $g_{MN}$, 5-form field strength $F_5$, and dilaton $\phi$ are activated. Then the equations of motion to be solved are

$$R_{MN} = \frac{1}{2}\partial_M\phi\partial_n\phi + \frac{1}{96}F_{MPQRS}F_N{}^{PQRS}\,, \tag{2.1}$$

$$\partial_M\left(\sqrt{g}g^{MN}\partial_N\phi\right) = 0\,. \tag{2.2}$$

The ansatz for the Janus solution takes the form

$$\begin{aligned} ds^2 &= f(\mu)\left(d\mu^2 + ds^2_{AdS_4}\right) + ds^2_{S^5}\,, \\ \phi &= \phi(\mu)\,, \\ F_5 &= 4f(\mu)^{5/2}d\mu\wedge d\omega_{AdS_4} + 4d\omega_{S^5}\,. \end{aligned} \tag{2.3}$$

The five sphere $S^5$ is intact, so is the SO(6) isometry. Since the $AdS_4$ space has the isometry SO(2, 3), the ansatz thus respects the SO(2, 3) $\times$ SO(6) isometry. When $\phi(\mu)$ is the constant, the scale factor $f(\mu)$ will be uniquely determined to be $1/\cos^2\mu$ and the geometry is the $AdS_4$ slicing of $AdS_5 \times S^5$. Once $\phi(\mu)$ starts varying, $f(\mu)$ will deviate

[2]Recently yet another possibility of the Janus-type solution was discussed in [19] concerning the holographic description of the cosmological backgrounds.

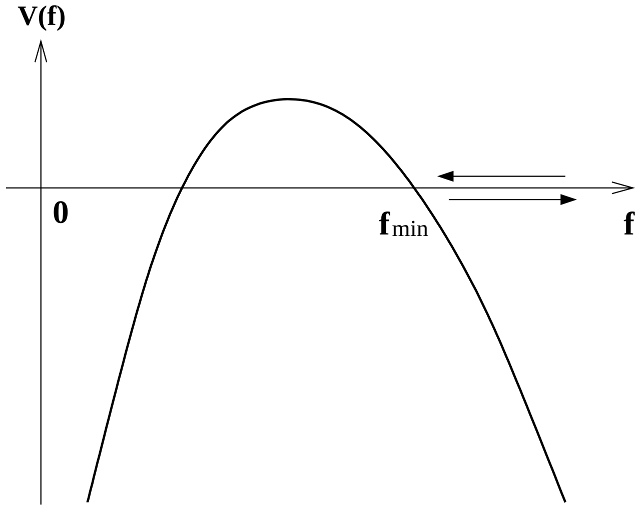


**Figure 1:** The motion of a particle governed by the Hamiltonian (2.6) with zero energy. This motion corresponds to a non-singular asymptotically AdS geometry. As $f$ goes to the infinity and the $c_0$ dependent effect of the nontrivial dilaton becomes negligible, the geometry approaches to the AdS space,



from $1/\cos^2\mu$, yielding the deformation of the AdS space. So in general the geometry can be viewed as a kind of $AdS_4$-sliced domain wall.

The equation of motion for the dilaton can be readily solved as

$$\phi(\mu)' = \frac{c_0}{f(\mu)^{3/2}}\,, \tag{2.4}$$

where the dash $'$ denotes the $\mu$-derivative. Then the Einstein equations yield

$$\begin{aligned} 2f'f' - 2ff'' &= -4f^3 + \frac{c_0^2}{2f}\,, \\ f'f' + 2ff'' + 12f^2 &= 16f^3\,. \end{aligned} \tag{2.5}$$

These are equivalent to the motion of a particle governed by the Hamiltonian

$$H(f,f') \equiv \frac{1}{2}f'f' + V(f) = \frac{1}{2}f'f' - \frac{1}{2}\left(4f^3 - 4f^2 + \frac{c_0^2}{6f}\right), \tag{2.6}$$

with zero energy $H(f,f') = 0$.

There are two possible branches of the solution specified by two distinct boundary conditions. In figure 1, the potential of the particle motion is depicted. The particle can reach either at $f = 0$ or $f = +\infty$. These two correspond to two different branches of the solution. However, the former turns out to be a singular solution, since the scalar curvature diverges at $f = 0$. So we will not be interested in this case. On the other hand, for the latter $f$ is bounded from below as shown in figure 1, and the corresponding geometry is singularity free. Furthermore as $f$ goes to the infinity, the $c_0$ dependent effect of the nontrivial dilaton becomes negiligible. Thus the geometry asymptotes to the AdS space and this is the solution we are interested in.

The equation $H(f,f')=0$ can be easily integrated to

$$\mu = \pm \int_{f_{\min}}^{f} \frac{d\tilde{f}}{2\sqrt{\tilde{f}^3 - \tilde{f}^2 + \frac{c_0^2}{24\tilde{f}}}} , \tag{2.7}$$

where $f_{\min}$ is the largest root of the rational function $P(x) = x^3 - x^2 + \frac{c_0^2}{24x}$, and we have set the origin of $\mu$ such that $\mu = 0$ at $f = f_{\min}$ for the convenience.

The dilaton equation can also be integrated to

$$\phi(\mu) = \phi_0 \pm \int_{f_{\min}}^{f(\mu)} \frac{c_0 d\tilde{f}}{2\tilde{f}^{3/2}\sqrt{\tilde{f}^3 - \tilde{f}^2 + \frac{c_0^2}{24\tilde{f}}}} , \tag{2.8}$$

where the choice of the sign is $+$ for $\mu \geq 0$ and $-$ for $\mu < 0$.

The analytic form of the function $f(\mu)$ was recently found in [18] in terms of the Weierstrass $\wp$-function. However here we will focus only on the qualitative features of the Janus solution.

First note that as we increase the value of the constant $c_0$, the potential $V(f)$ goes down. It is easy to show that at $c_0 = 9/4\sqrt{2}$ the top of the potential is at $V(f) = 0$, in other words two roots of the rational function $P(x)$ coalesce. Beyond this point, the particle coming in from $f = +\infty$ always reaches at $f = 0$ in the end. So the solution becomes singular for $c_0 > 9/4\sqrt{2}$. Hence we will only consider the case

$$0 \leq c_0 \leq 9/4\sqrt{2} . \tag{2.9}$$

Second, since the constant $\mu$ corresponds to the $AdS_4$ slicing of the (deformed) $AdS_5$, it ranges from $-\pi/2$ to $\pi/2$ in the undeformed case $c_0 = 0$. In general $\mu$ is a monotically increasing function of $c_0$, as one can easily see it from the expression (2.7), so its range is

$$-\mu_0 \leq \mu \leq \mu_0 \qquad \left(\mu_0 \geq \frac{\pi}{2}\right) . \tag{2.10}$$

Indeed $\mu_0$ diverges when $c_0$ approaches the critical value $9/4\sqrt{2}$, since it takes an infinite time for a particle to get to $f = f_{\min}$ when the top of the potential is precisely level with $V = 0$.

Third, the dilaton approaches the constants at $\mu = \pm\mu_0$, as $f = +\infty$ there and the solution asymptotes to $AdS_5 \times S^5$. In fact the dilaton takes different values, $\phi = \phi_0 - \Delta\phi_0$ at $\mu = -\mu_0$ and $\phi = \phi_0 + \Delta\phi_0$ at $\mu = \mu_0$, as one can see it from

$$2\Delta\phi_0 = \phi(\mu_0) - \phi(-\mu_0) = 2\int_{f_{\min}}^{\infty} \frac{c_0 df}{2f^{3/2}\sqrt{f^3 - f^2 + \frac{c_0^2}{24f}}} > 0 . \tag{2.11}$$

Finally the Janus solution has a peculiar structure at the boundary of the deformed $AdS_5$. This is where its name comes from. To see it, let us first consider the undeformed $AdS_5$ space. The $AdS_4$ slicing can be expressed as

$$ds^2 = \frac{1}{\cos^2\mu}\left(d\mu^2 + ds^2_{AdS_4}\right) , \tag{2.12}$$

where $-\pi/2 \le \mu \le \pi/2$. For the global $AdS_4$ coordinate, it takes the form

$$ds^2 = \frac{1}{\cos^2\mu\cos^2\eta}\left(-d\tau^2 + \cos^2\eta d\mu^2 + d\eta^2 + \sin^2\eta d\Omega_2^2\right) , \tag{2.13}$$

where $0 \le \eta \le \pi/2$, and for the $AdS_4$ Poincare patch,

$$ds^2 = \frac{1}{y^2\cos^2\mu}\left(-dt^2 + d\vec{x}_2^2 + dy^2 + y^2 d\mu^2\right) , \tag{2.14}$$

where $y \ge 0$.

The former (2.13) can be transformed to the global $AdS_5$ coordinate by

$$\tan\phi = \frac{\tan\eta}{\sin\mu} , \qquad \cos\theta = \cos\mu\cos\eta , \tag{2.15}$$

where $0 \le \phi \le \pi$ and $0 \le \theta \le \pi/2$. In terms of $\phi$ and $\theta$ the metric becomes

$$ds^2 = \frac{1}{\cos^2\theta}\left(-d\tau^2 + d\theta^2 + \sin^2\theta\left(d\phi^2 + \sin^2\phi d\Omega_2^2\right)\right) . \tag{2.16}$$

The boundary of $AdS_5$ is at $\theta = \pi/2$ which corresponds to $\mu = \pm\pi/2$ or $\eta = \pi/2$. At $\mu = \pm\pi/2$, $\tan\phi = \pm\tan\eta$ respectively. Since the range of $\eta$ is from 0 to $\pi/2$, $\tan\phi > 0$ for $\mu = +\pi/2$ and $\tan\phi < 0$ for $\mu = -\pi/2$. This means that $0 \le \phi < \pi/2$ for $\mu = +\pi/2$ and $\pi/2 < \phi \le \pi$ for $\mu = -\pi/2$. Hence $\mu = +\pi/2$ corresponds to the upper hemi-sphere and $\mu = -\pi/2$ to the lower hemi-sphere of $S^3$, and they are joined at $\eta = \pi/2$.

Qualitatively the same is true for the deformed case. The only difference is that $\mu$ now ranges from $-\mu_0$ to $+\mu_0$ with $\mu_0 > \pi/2$. Thus $\mu = +\mu_0$ corresponds to the upper hemi-sphere $S^3_+$ and $\mu = -\mu_0$ to the lower hemi-sphere $S^3_-$ of $S^3$, and they are joined at $\eta = \pi/2$. This is depicted in figure 2.

Similarly for the latter the coordinate transformation

$$x = y\sin\mu , \qquad z = y\cos\mu , \tag{2.17}$$

brings the metric (2.14) into

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + d\vec{x}_2^2 + dx^2 + dz^2\right) . \tag{2.18}$$

The boundary of $AdS_5$ is at $z = 0$ which corresponds to $\mu = \pm\pi/2$ or $y = 0$. At $\mu = \pm\pi/2$, $x = \pm y$ respectively. Since $y \ge 0$, $x > 0$ for $\mu = +\pi/2$ and $x < 0$ for $\mu = -\pi/2$ except at $y = 0$. Hence $\mu = \pm\pi/2$ each corresponds to a half $\mathbb{R}^{3,1}_\pm$ of the boundary $\mathbb{R}^{3,1}$, and they are joined at $y = 0$.

Again qualitatively the same is true for the deformed case. Only difference from the undeformed case is the range of $\mu$. This is shown in figure 3.

This peculiar structure of the Janus geometry reveals an interesting feature from the viewpoint of the AdS/CFT duality [1]. Since the constant value of the dilaton differs in each half of the boundary at $\mu = \pm\mu_0$, the SYM coupling jumps when it moves from one half of the space to another, yielding two different values of the coupling constants dividing the spacetime into two. That is, it is as if the dual gauge theory had two faces — hence



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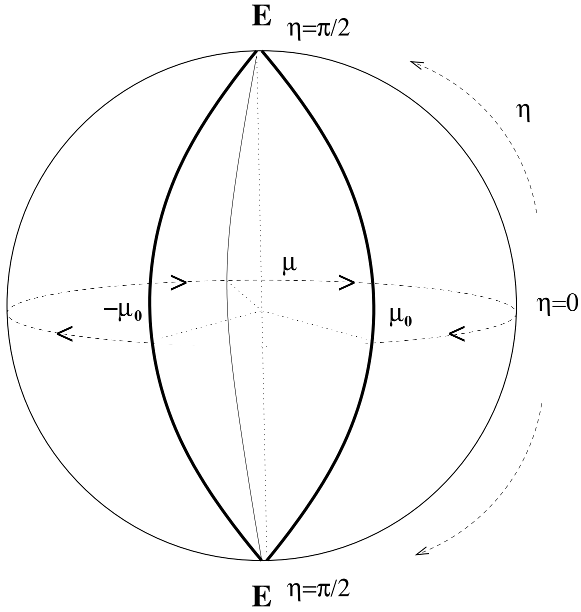



**Figure 2:** The conformal mapping of the spatial section of Janus in the global coordinates. The two dimensional surface shown is parametrized by $(\mu, \eta)$, and has the topology of disk. Each point in the upper half of the surface is identified with that in the lower half due to the rotational symmetry associated with $S^2$ whose $(\text{radius})^2 \propto \sin^2 \eta$. The boundary is indicated by the thick line corresponding to $S^3 = S^3_+(\mu = \mu_0) \cup S^2(\text{E}) \cup S^3_-(\mu = -\mu_0)$, where E denotes the equator of $S^3$.

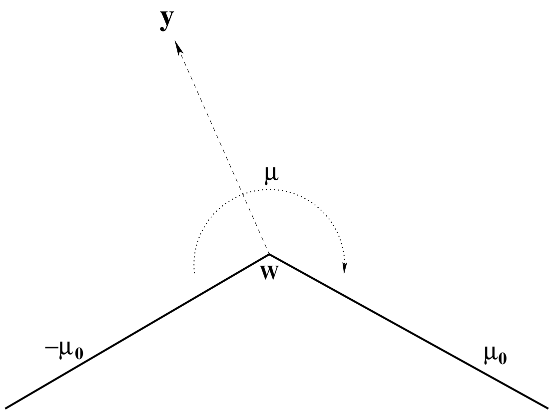



**Figure 3:** The conformal mapping of the spatial section of Janus in the Poincare patch. The plane is parametrized by $(\mu, y)$. Each point on the plane corresponds to $\mathbb{R}^2$. The boundary is indicated by the wedge with thick line corresponding to $\mathbb{R}^3 = \mathbb{R}^3_+(\mu = \mu_0) \cup \mathbb{R}^2(\text{W}) \cup \mathbb{R}^3_-(\mu = -\mu_0)$.

the name Janus, the god of gates, doors, doorways, beginnings, and endings in Roman mythology who is often symbolized by two faces. Two faces are joined at the interface. Although the 4-dimensional conformal symmetry SO$(2, 4)$ is partially broken, the conformal symmetry SO$(2, 3)$ on the interface is preserved. Hence the dual gauge theory is thought of as an interface CFT [11].

## 3. The nested Janus

Since the $AdS_4$-slice can be further sliced into $AdS_3$ and even further into $AdS_2$, it may be possible to nest little Janus into the interface of the parent Janus in a self-similar way. In terms of the dual gauge theory, this would correspond to having an interface CFT of higher codimensions. In this section, we will show that this is indeed the case. It turns out that the result is as simple as it could be.

The ansatz for the nested Janus solution takes the form

$$\begin{aligned} ds^2 &= f_1(\mu_1)\left(d\mu_1^2 + f_2(\mu_2)\left(d\mu_2^2 + f_3(\mu_3)\left(d\mu_3^2 + ds^2_{AdS_2}\right)\right)\right) + ds^2_{S^5}\,, \\ \phi &= \phi_1(\mu_1) + \phi_2(\mu_2) + \phi_3(\mu_3)\,, \\ F_5 &= 4f_1(\mu_1)^{5/2} f_2(\mu_2)^2 f_3(\mu_3)^{3/2} d\mu_1 \wedge d\mu_2 \wedge d\mu_3 \wedge d\omega_{AdS_2} + 4 d\omega_{S^5}\ . \end{aligned} \tag{3.1}$$

The isometry of the geometry is SO(2, 1) × SO(6) in generic cases. It could be enhanced to larger isometries in special cases.

The equation of motion for the dilaton yields

$$f_2^2 f_3^{3/2}\left(f_1^{3/2}\phi_{1,1}\right)_{,1} + f_1^{3/2} f_3^{3/2}\left(f_2\phi_{2,2}\right)_{,2} + f_1^{3/2} f_2\left(f_3^{1/2}\phi_{3,3}\right)_{,3} = 0\,, \tag{3.2}$$

where the symbol $h_{,i}$ for any function $h$ denotes the derivative of $h$ with respect to $\mu_i$. It can be solved by

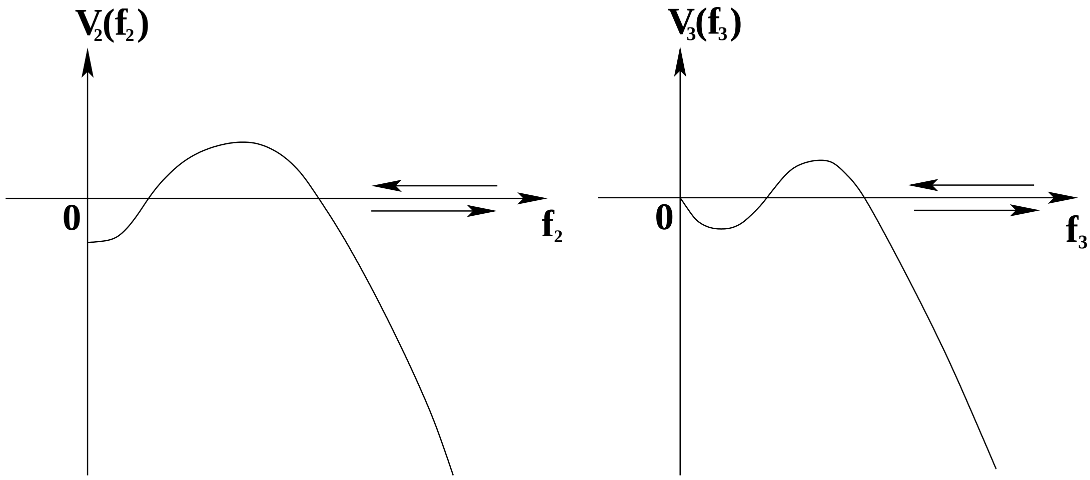
$$\phi_{1,1} = \frac{c_1}{f_1^{3/2}}\,, \qquad \phi_{2,2} = \frac{c_2}{f_2}\,, \qquad \phi_{3,3} = \frac{c_3}{f_3^{1/2}}\ . \tag{3.3}$$

The Einstein equations then yield

$$\begin{aligned} 2f_{1,1}f_{1,1} - 2f_1 f_{1,1,1} &= -4f_1^3 + \frac{c_1^2}{2f_1}\,, \\ -\frac{f_{1,1}f_{1,1} + 2f_1 f_{1,1,1}}{f_1^2} + \frac{6f_{2,2}f_{2,2} - 6f_2 f_{2,2,2}}{f_2^3} &= -16f_1 + \frac{2c_2^2}{f_2^3}\,, \\ -\frac{f_{1,1}f_{1,1} + 2f_1 f_{1,1,1}}{f_1^2} - \frac{2f_{2,2,2}}{f_2^2} + \frac{4f_{3,3}f_{3,3} - 4f_3 f_{3,3,3}}{f_2 f_3^3} &= -16f_1 + \frac{2c_3^2}{f_2 f_3^2}\,, \\ -\frac{f_{1,1}f_{1,1} + 2f_1 f_{1,1,1}}{f_1^2} - \frac{2f_{2,2,2}}{f_2^2} + \frac{f_{3,3}f_{3,3} - 2f_3 f_{3,3,3} - 4f_3^2}{f_2 f_3^3} &= -16f_1\ . \end{aligned}$$

By inspection it turns out that these equations can be solved by

$$\begin{aligned} f_{1,1}f_{1,1} + 2f_1 f_{1,1,1} + 12f_1^2 - 16f_1^3 &= 0\,, \\ f_2 f_{2,2,2} + 4f_2^2 - 6f_2^3 &= 0\,, \\ f_{3,3}f_{3,3} - 2f_3 f_{3,3,3} - 4f_3^2 + 8f_3^3 &= 0\,, \\ 2f_{1,1}f_{1,1} - 2f_1 f_{1,1,1} + 4f_1^3 - \frac{c_1^2}{2f_1} &= 0\,, \\ 3f_{2,2} - 2f_2 f_{2,2,2} + 4f_2^2 - 6c_2^2 &= 0\,, \\ 3f_{3,3}f_{3,3} - 2f_3 f_{3,3,3} + 4f_3^2 - 2c_3^2 f_3 &= 0\ . \end{aligned}$$




(A) (B)

**Figure 4:** The motions of a particle governed (A) by the Hamiltonian (3.5) and (B) by (3.6) with zero energy, corresponding to the $AdS_3$ and $AdS_2$-sliced Janus respectively.

Note that three scale factors $f_{i=1,2,3}$ do not mix in the equations. Similarly to the Janus solution reviewed in the previous section, it is easy to show that these equations are equivalent to the particle motion governed by the Hamiltonians

$$H_1(f_1, f_{1,1}) \equiv \frac{1}{2} f_{1,1} f_{1,1} + V_1(f_1) = \frac{1}{2} f_{1,1} f_{1,1} - \frac{1}{2}\left(4f_1^3 - 4f_1^2 + \frac{c_1^2}{6f_1}\right) , \tag{3.4}$$

$$H_2(f_2, f_{2,2}) \equiv \frac{1}{2} f_{2,2} f_{2,2} + V_2(f_2) = \frac{1}{2} f_{2,2} f_{2,2} - \frac{1}{2}\left(4f_2^3 - 4f_2^2 + 2c_2^2\right) , \tag{3.5}$$

$$H_3(f_3, f_{3,3}) \equiv \frac{1}{2} f_{3,3} f_{3,3} + V_3(f_3) = \frac{1}{2} f_{3,3} f_{3,3} - \frac{1}{2}\left(4f_3^3 - 4f_3^2 + c_3^2 f_3\right) , \tag{3.6}$$

with zero energies $H_i(f_i, f_{i,i}) = 0$ $(i = 1, 2, 3)$.

The scale factor $f_1(\mu_1)$ obeys exactly the same equation as that of the $AdS_4$-sliced Janus in the previous section, corresponding to the particle motion depicted in figure 1. More interestingly the scale factors $f_2(\mu_2)$ and $f_3(\mu_3)$ obey respectively the equation for the $AdS_2$ and $AdS_3$-sliced Janus generalized in [10], corresponding to the particle motions shown in figure 4.

Hence the geometry represents Janus within Janus — $AdS_2$-sliced Janus $\subset$ $AdS_3$-sliced Janus $\subset$ $AdS_4$-sliced Janus. Little Janus inside the parent Janus follows the remarkably simple "nesting" rule in a self-smilar fashion, and the dilaton follows the simple sum rule.

As a further illustration, let us consider a simplified example — the $AdS_4$ slicing of the undeformed $AdS_5$ where the $AdS_4$-slice is further sliced by $AdS_3$. In the Poincare patch the metric takes the form

$$ds^2 = \frac{1}{\cos^2 \mu_1}\left(d\mu_1^2 + \frac{1}{y_2^2 \cos^2 \mu_2}\left(-dt^2 + dx_3^2 + dy_2^2 + y_2^2 d\mu_2^2\right)\right) , \tag{3.7}$$

where $-\pi/2 \leq \mu_{1,2} \leq \pi/2$ and $y_2 \geq 0$.

We will transform it to the standard form of the Poincare patch for $AdS_5$ in two steps.

In the first step, we introduce the new coordinates by

$$x_2 = y_2 \sin\mu_2 \, , \qquad y_1 = y_2 \cos\mu_2 \, . \tag{3.8}$$

This brings (3.7) into the form

$$ds^2 = \frac{1}{\cos^2\mu_1}\left(d\mu_1^2 + \frac{1}{y_1^2}\left(-dt^2 + dx_3^2 + dx_2^2 + dy_1^2\right)\right) \, , \tag{3.9}$$

where $y_1 \geq 0$.

Similarly in the next step, by the coordinate transformation

$$x_1 = y_1 \sin\mu_1 \, , \qquad z = y_1 \cos\mu_1 \, , \tag{3.10}$$

the metric (3.9) yields

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + dx_3^2 + dx_2^2 + dx_1^2 + dz^2\right) \, , \tag{3.11}$$

where $z \geq 0$.

The coordinate transformation we made from $(t, x_3, \mu_1, \mu_2, y_2)$ to $(t, x_3, x_2, x_1, z)$ is

$$z = y_2 \cos\mu_1 \cos\mu_2 \, , \qquad x_1 = y_2 \sin\mu_1 \cos\mu_2 \, , \qquad x_2 = y_2 \sin\mu_2 \, . \tag{3.12}$$

Now the boundary of $AdS_5$ is at $z = 0$ which corresponds to $\mu_1 = \pm\pi/2$, $\mu_2 = \pm\pi/2$, or $y_2 = 0$. At $\mu_2 = \pm\pi/2$, $x_1 = 0$ and $x_2 = \pm y_2$ respectively. Since $y_2 \geq 0$, $x_2 > 0$ for $\mu_2 = +\pi/2$ and $x_2 < 0$ for $\mu_2 = -\pi/2$ except at $y_2 = 0$. Hence $\mu_2 = \pm\pi/2$ each corresponds to a half $\mathbb{R}^{2,1}_{\pm}$ of the codimension 1 subspace $\mathbb{R}^{2,1}$ (defined by $x_1 = 0$) of the boundary $\mathbb{R}^{3,1}$, and they are joined at $y_2 = 0$ which is the codimension 2 subspace $\mathbb{R}^{1,1}$ (defined by $x_1 = x_2 = 0$). Similarly at $\mu_1 = \pm\pi/2$, $x_1 = \pm y_2 \cos\mu_2$ respectively. Since $y_2 \geq 0$ and $-\pi/2 \leq \mu_2 \leq \pi/2$, $x_1 > 0$ for $\mu_1 = \pi/2$ and $x_1 < 0$ for $\mu_1 = -\pi/2$ except at $y_2 = 0$ or $\mu_2 = \pm\pi/2$. Hence $\mu_1 = \pm\pi/2$ each corresponds to a half $\mathbb{R}^{3,1}_{\pm}$ of the boundary $\mathbb{R}^{3,1}$ joined at the subspace $\mathbb{R}^{2,1}$ defined by $y_2(\mu_2 + \pi/2)(\mu_2 - \pi/2) = 0$.

As in the simpler Janus in the previous section, qualitatively the same is true for the deformed case. Only difference is the range of $\mu_{1,2}$, instead of $-\pi/2 \leq \mu_{1,2} \leq \pi/2$, we have $-\mu_{1,0} \leq \mu_1 \leq \mu_{1,0}$ and $-\mu_{2,0} \leq \mu_2 \leq \mu_{2,0}$ where $\mu_{i,0} \geq \pi/2$.

It should also be clear that the $AdS_3$-sliced Janus nests in the codimension 1 subspace $\mathbb{R}^{2,1}$ which is nothing but the interface of the $AdS_4$-sliced Janus — hence the name nested Janus. This structure is depicted in figure 5. It is straightforward to extend this argument further to the $AdS_2$ slicing.[3] So we will not repeat it here.

A few remarks are in order: First, the equations $H_i(f_i, f_{i,i}) = 0$ $(i = 1, 2, 3)$ as well as the dilaton equations (3.2) can be easily integrated as in the case of the simple Janus in the previous section. As mentioned before, the analytic form of the function $f_1(\mu_1)$ was

[3] In the case of the global coordinates, the boundary of $AdS_2$ consists of two disconnected components corresponding to $\eta = \pm\pi/2$ in (2.13) if applied to the $AdS_2$ space *per se*. However, since $\eta$ ranges from 0 to $\pi/2$ for the $AdS_2$ slice in $AdS_5$, one of the components is singled out for the $AdS_2$-sliced Janus. We would like to thank M. Gutperle for asking us to clarify this point.

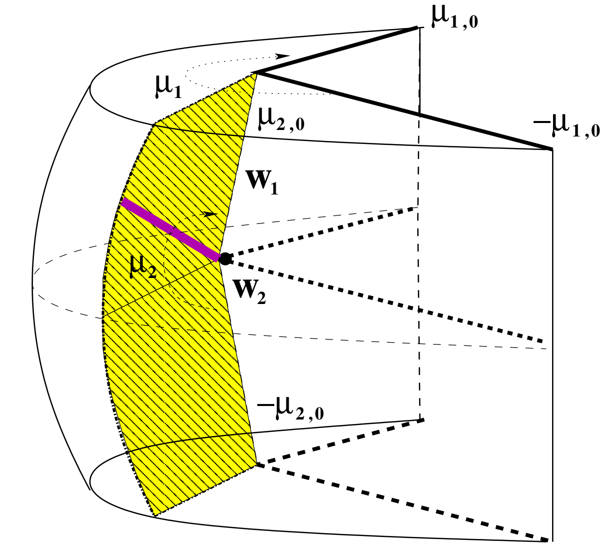



**Figure 5:** The $AdS_4$-sliced Janus and $AdS_3$-sliced Janus within it in the Poincare patch. The patterned yellow area is a (constant $\mu_1$) slice of the (deformed) $AdS_4$. The thick magenta line in it is a (constant $\mu_2$) slice of the (deformed) $AdS_3$. The wedge $W_1$ depicted as a vertical line is $\mathbb{R}^{2,1} = \mathbb{R}^{2,1}_+ \cup \mathbb{R}^{1,1} \cup \mathbb{R}^{2,1}_-$ which is the interface of two faces of the $AdS_4$-sliced Janus. The wedge $W_2$ depicted as a point is $\mathbb{R}^{1,1} = \mathbb{R}^{1,1}_+ \cup \mathbb{R}^1 \cup \mathbb{R}^{1,1}_-$ corresponding to the interface of two faces of the $AdS_3$-sliced Janus nested in the $AdS_4$-sliced Janus. Upon the inclusion of the $AdS_2$-sliced Janus, the timelike $\mathbb{R}^1 \subset \mathbb{R}^{1,1}$ would correspond to its interface.

recently found in [18] in terms of the Weierstrass $\wp$-function. That of $f_3(\mu_3)$ is simpler and the expicit form is given in [10] in terms of the Jacobi elliptic function. On the other hand, that of $f_2(\mu_2)$ is yet to be found.

Regardless of the explicit form of the scale factors, the qualitative feature of each $AdS_{4,3,2}$-sliced Janus is essentially all the same — their boundaries consist of two parts joined at the interface, and the constant value of the dilaton differs in each half of the boundary at $\mu_i = \pm\mu_{i,0}$ where $\phi_i(\pm\mu_{i,0})$ takes the value $\phi_{i,0} \pm \Delta\phi_{i,0}$ respectively.

Second, as in the case of the $AdS_4$-sliced Janus, there are upper bounds on the constants $c_{2,3}$ for the $AdS_{2,3}$-sliced Janus, above which the non-singular solution ceases to exist. As can be easily seen from the form of the potential in (3.5) and (3.6), as we increase the value of $c_{2,3}$, the potentials go down, and at some values of $c_{2,3}$ the top of the potentials is leveled with $V_{2,3} = 0$. It is easy to find that it occurs at $c_2 = (2/3)^{3/2}$ and $c_3 = 1$. Hence we are only interested in the value of $c_{1,2,3}$ in the ranges

$$\begin{aligned} 0 \le c_1 &\le 9/4\sqrt{2}\,, \\ 0 \le c_2 &\le (2/3)^{3/2}\,, \\ 0 \le c_3 &\le 1\,. \end{aligned}$$

Third, it is clear that, for example, the $AdS_2$-sliced Janus alone can exist in $AdS_5$ without nesting. This is the codimension 3 Janus. Similarly the $AdS_3$-sliced Janus alone corresponds to the codimension 2 Janus. Also any combination of two is obviously possible.

Finally the boundary of the nested Janus is at $f_i(\mu_i) = +\infty$ $(i = 1, 2, 3)$ or equivalently at $\mu_i = \pm\mu_{i,0}$ where the geometry asymptotes to $AdS_5$. The asymptotics of the nested Janus solution can be easily found as

$$f_i(\mu_i) \sim \frac{1}{(\mu_i \mp \mu_{i,0})^2}\,, \tag{3.13}$$

$$\begin{aligned}\phi(\mu_1, \mu_2, \mu_3) \sim\ & \phi_0 \pm \Delta\phi_{1,0} \mp \frac{c_1}{4}(\mu_1 \mp \mu_{1,0})^4 \\ & \pm\Delta\phi_{2,0} + \frac{c_2}{3}(\mu_2 \mp \mu_{2,0})^3 \pm \Delta\phi_{3,0} \mp \frac{c_3}{2}(\mu_3 \mp \mu_{3,0})^2\,,\end{aligned} \tag{3.14}$$

where we have defined $\phi_0 = \sum_{i=1,3} \phi_{i,0}$.

In the dual gauge theory, as proposed in [1] and refined in [11], the SYM coupling discontinuously jumps when it crosses the interface of two halves of the plane. In the nested Janus case, the interface accommodates lower dimensional interfaces within itself, so there is a sequence of jumps of the gauge coupling. We propose that the deformation to add to $\mathcal{N} = 4$ SYM is

$$\Delta S = -\int d^4x \Big(\Delta\gamma_1 \varepsilon(x_1) + \Delta\gamma_2 \delta(x_1)\varepsilon(x_2) + \Delta\gamma_3 \delta(x_1)\delta(x_2)\varepsilon(x_3)\Big)\mathcal{L}_{\rm SYM}\,, \tag{3.15}$$

where $\mathcal{L}_{\rm SYM}$ is the Lagrangian density of $\mathcal{N} = 4$ SYM with the difference by a total derivative term [11], and the step function $\varepsilon(x) = 2\theta(x) - 1$. The undeformed action is given by $\frac{1}{g^2_{\rm YM}}\mathcal{L}_{\rm SYM}$ with $1/g^2_{\rm YM} = \prod_{i=1}^3 (1/g_i^2) \equiv \prod_{i=1}^3 \left(\frac{1}{2}(1/g^2_{i+} + 1/g^2_{i-})\right)$ where $g^2_{i\pm} = 2\pi e^{\phi_{i,0} \pm \Delta\phi_{i,0}}$. The deformation parameters $\Delta\gamma_i$ are related to $g_{i\pm}$s by

$$\Delta\gamma_i = \frac{1}{g^2_{\rm YM}} \frac{g^2_{i+} - g^2_{i-}}{g^2_{i+} + g^2_{i-}}\,. \tag{3.16}$$

This deformation will induce the vev for the dimension 4 operator $\mathcal{L}_{\rm SYM}$, as expected from the subleading contribution in the asymptotic expansion of the dilaton.

To see it, let us take a closer look at the asymptotic expansion (3.14) of the dilaton. Since we wish to consider the dual gauge theory on $\mathbb{R}^{3,1}$, we need to know the relation between two coordinate systems, the AdS-slicings $(t, \mu_1, \mu_2, \mu_3, y_3)$ and the $AdS_5$ Poincare patch $(t, x_1, x_2, x_3, z)$, where $(t, y_3)$ is the coordinates of the $AdS_2$ Poincare patch. Near $\mu_i = \pm\mu_{i,0}$ the nested Janus is approximately $AdS_5$, so similarly to the transformation (3.12) discussed above, we can find that near the boundary

$$\begin{aligned} z &\sim \pm y_3 \sin(\mu_1 \pm \mu_{1,0}) \sin(\mu_2 \pm \mu_{2,0}) \sin(\mu_3 \pm \mu_{3,0})\,, \\ x_1 &\sim \mp y_3 \cos(\mu_1 \pm \mu_{1,0}) \sin(\mu_2 \pm \mu_{2,0}) \sin(\mu_3 \pm \mu_{3,0})\,, \\ x_2 &\sim -y_3 \cos(\mu_2 \pm \mu_{2,0}) \sin(\mu_3 \pm \mu_{3,0})\,, \\ x_3 &\sim \mp y_3 \cos(\mu_3 \pm \mu_{3,0})\,, \end{aligned}$$

where the order of signs are correlated. We have fixed the signs as follows: First recall that $\mu_i$s are in the ranges $-\mu_{i,0} \leq \mu_i \leq \mu_{i,0}$ and we are considering $\mu_i \sim \pm\mu_{i,0}$. So we have $\sin(\mu_i + \mu_{i,0}) > 0$, $\sin(\mu_i - \mu_{i,0}) < 0$, and $\cos(\mu_i \pm \mu_{i,0}) > 0$. Now since $z \geq 0$, that fixes the sign in the first line. The rest is fixed based on the fact that $x_i > 0$ at $\mu_i = \mu_{i,0}$ and $x_i < 0$ at $\mu_i = -\mu_{i,0}$.

We can then deduce that

$$\tan(\mu_1 \pm \mu_{1,0}) \sim -\frac{z}{x_1}\,,$$
$$\tan(\mu_2 \pm \mu_{2,0}) \sim \pm\sqrt{\frac{z^2+x_1^2}{x_2^2}}\,,$$
$$\tan(\mu_3 \pm \mu_{3,0}) \sim \pm\sqrt{\frac{z^2+x_1^2+x_2^2}{x_3^2}}\;.$$

Hence we obtain

$$(\mu_1 \pm \mu_{1,0})^4 \sim \left(\frac{z}{x_1}\right)^4, \tag{3.17}$$

$$\begin{aligned}(\mu_2 \pm \mu_{2,0})^3 &\sim \pm\left(\frac{z^2+x_1^2}{x_2^2}\right)^{3/2} = \pm\left(\frac{z}{z^2+x_1^2}\right)\frac{z^2+x_1^2}{z}\left(\frac{z^2+x_1^2}{x_2^2}\right)^{3/2}\\ &\sim \pm\pi\delta(x_1)\frac{z^2+x_1^2}{z}\left(\frac{z^2+x_1^2}{x_2^2}\right)^{3/2} \sim -\pi\delta(x_1)\frac{z^4}{x_2^3}\,,\end{aligned} \tag{3.18}$$

$$\begin{aligned}(\mu_3 \pm \mu_{3,0})^2 &\sim \frac{z^2+x_1^2+x_2^2}{x_3^2} = \left(\frac{z^2}{(z^2+x_1^2+x_2^2)^2}\right)\frac{(z^2+x_1^2+x_2^2)^2}{z^2}\frac{z^2+x_1^2+x_2^2}{x_3^2}\\ &\sim \pi\delta(x_1)\delta(x_2)\frac{(z^2+x_1^2+x_2^2)^2}{z^2}\frac{z^2+x_1^2+x_2^2}{x_3^2} \sim \pi\delta(x_1)\delta(x_2)\frac{z^4}{x_3^2}\;.\end{aligned} \tag{3.19}$$

Strictly speaking, the order of limits we prescribed here needs to be justified more rigorously. However, the result so obtained appears to make sense from the viewpoint of AdS/CFT.

Therefore according to the AdS/CFT dictionary [20], what we would expect is

$$\langle \mathcal{L}_{\rm SYM}\rangle = \varepsilon(x_1)\frac{N^2}{2\pi^2}\frac{c_1}{4x_1^4} + \delta(x_1)\varepsilon(x_2)\frac{N^2}{2\pi}\frac{c_2}{3|x_2|^3} + \delta(x_1)\delta(x_2)\varepsilon(x_3)\frac{N^2}{2\pi}\frac{c_3}{2x_3^2}\,, \tag{3.20}$$

generalizing the result of [11, 1] to the case of nonvanishing $c_2$ and $c_3$.

## 4. Discussions

What makes Janus potentially interesting is its remarkable simplicity. Here we have seen that this trait persists in the generalization we discussed. The simple "nesting" rule found here is reminiscent of the intersection rule for the supersymmetric brane configurations. So this might be related to the fact that the Janus solution has the fake supersymmetry [10, 21] even though it is not a supersymmetric geometry in the standard sense.[4]

The stability of the nested Janus solution needs to be examined. However, since the $AdS_d$-sliced Janus for any dimension $d$ was shown to be stable against a large class of perturbations [10], we believe that this is also the case for the nested Janus which is made up of a sequence of the $AdS_d$-sliced Janus each one of which is stable.

[4]The "fake" supersymmetry constraint on the AdS domain wall was first derived in [14], and its relation to the real supergravity was estabilished in [22].



JHEP05(2006)031

It is worthwhile to study the dual interface CFT and compare the results with the supergravity predictions, generalizing the firm analysis carried out in [11]. The holographic renormalization group method developed in [23] enables more efficient computations of the correlation functions. In this paper we computed only the vev of a particular dimension 4 operator, but it is worth calculating the correlation functions by using their method to check with the interface CFT expectation. In particular, since the $AdS_2$-sliced Janus is simpler relative to the higher dimensional counterparts, it might be useful to study this case more in details.

Finally it must be possible to find the supersymmetric version of the nested Janus. In the dual interface CFT side, there must exist the interface interactions which restore the supersymmetries [11, 12]. Perhaps the thorough classification, as was done in the case of the supersymmetric Janus in [12], of such interactions in this more general case would in turn suggest whether and how the supersymmetrization can be made in type IIB supergravity.

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